On the absence of zeros of ς(z) out of Re(z)=1/2.

Introduction.

The Riemann hypothese –RH- is considered to be one of the most relevant problems still unsolved. Bombieri's statement [1] is very accurate concerning its scope. Thought many attempts achieved to demonstrate the existence of zeros in the so-called "critical line", really the existence of infinite –isolated, thus numerable- in this line is due to Hardy [2] long time ago. Nevertheless, the non existence of zeros out of the critical line remains undemonstrated.

In this contribution, we show that only in the critical line the existence of ceros for the Riemann function is true and confirm the topological character of these zeros as numerables following a pure analytic functions methodology.

Riemann conjecture.

In his famous book of 1748, Euler proved what is now named the Euler product formula [3]. This product is the result of the infinite sum:

$$\sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_{\{P\}}(1-\frac{1}{p^s})^{-1} \qquad \text{for any integer variable } s\text{>1}$$

where {P} is the infinite set of primes.

Riemann extended Euler's result to continue analytically in "s" and he stablished the Functional Relation:

$$\Lambda(s) \coloneqq \pi^{-\frac{s}{2}}\,\Gamma\left(\frac{S}{2}\right)\,\varsigma(s) = \Lambda(s-1),$$

Γ being the Gamma function. RH is the assertion that all the zeros of $\Lambda(s)$ are on the line in the complex plane Re(s)=$\frac{1}{2}$ As for Example, P.Sarnak [4] has claimed "…elegant, crispy, falsifiable and far-reaching, this conjecture is the epitome of what a good conjecture should be…".Moreover, its generalizations to other zeta functions, number theory and mathematical physics applications have many striking consequences, making the conjecture even more relevant.

In 1914, Hardy demonstrated the existence of infinite zeros in Re(s)=$\frac{1}{2}$ [2], and later in 1989 it was demonstrated that more of 2/5 of those zeros lied in Re(s)=$\frac{1}{2}$.[5]. Nevertheless, the question about the absolute non existence of zeros out Re(s)=$\frac{1}{2}$ is still remaning.

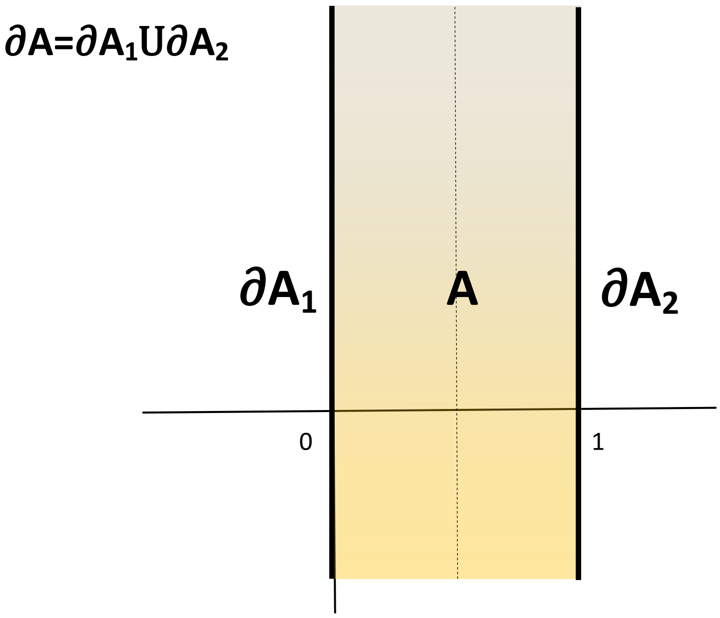
We will show through an analytical approximation that Riemann conjecture is true, while confirming simultaneously that the zeros are infinite numerable.

A new function (z) without z=1 as pole.

Let us remove the z = 1 pole in $\Lambda(s)$ while maintaining its symmetry under "z→1-z" by defining the function (z):

$$\xi(\mathrm{z}) := \mathrm{z}(\mathrm{z}-1)\Lambda(s) = \mathrm{z}(\mathrm{z}-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\varsigma(s) = \xi(1-\mathrm{z})$$

This function is analytic having now zeros in z=0 and z=1. In the boundaries of the critical region, being eliminated the poles the function $\xi(\mathrm{z})$ is analytic.



Region of analicity for $\xi(\mathrm{z})$. Interior of A is open and adh(A) is convex. Doted line represents the "critical line", Re(z)=1/2.

Thus, we have the following analytical function in A, open and simply connected ⊂ℂ:

$$\xi(\mathrm{z}) := \mathbb{C} - - - - - - - - - - - - - \rightarrow \mathbb{C}$$

$$x + iy - - - - - - - - - - - - \rightarrow u(x,y) + iv(x,y)$$

Cauchy-Riemann conditions being satisfied: $\partial_x u = \partial_y v \ \ and \ \partial_y u = -\partial_x v$. Consequently, both functions are harmonic in an open domain whose boundary has a regular function "f" defined.

Thus, for each harmonic function $(u(x,y), v(x,y))$ we identify an equivalent Dirichlet condition. For $u(x,y)$:

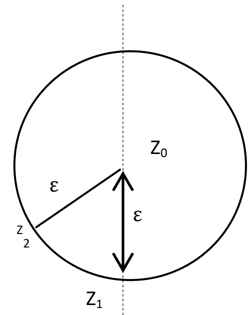
$$\Delta u(x,y) = 0,\ u(x,y) = f(x,y), analytic\ in\ \partial A.$$

Similar for $v(x,y)$.The existence for an analytical solution unique is shown specific for a convex domain. Undergraduate texts of the Dirichlet problem can be found elsewhere.

The non existence of zeros of (z) out of Re(z)=1/2.

First, let´s point out that since ξ(z) is analytic in adh(A)⊂ℂ, so the zeros are isolated points due to identity principle (non accumulated points in the subset “zeros of ξ(z)”. In consequence, the zeros of $u(x,y)$ are also isolated. Same for $v(x,y)$.

Let $z_0 = \left(\frac{1}{2}, y_o\right), z_1 = \left(\frac{1}{2}, y_1\right)$ two consecutive zeros for ξ(z) coming from a zero of $\varsigma(s)$ in the “critical line”. As isolated points, the identification of a distance between them is possible; let “ε” be that distance $d(z_1, z_0)$. Be $\overline{\mathrm{B}(\mathrm{z_o}, \varepsilon)}$ the ball defined by the Taylor series around z_0, as ξ(z) is analytical and unique, as stated previously.



Schema of vicinity in z_0, z_1 as a zeros of ξ(z)

Assuming z_0 a k-zero, the Taylor expansion is:

$$\xi(\mathrm{z}) = (\mathrm{z} - \mathrm{z_0})^{\mathrm{k}}\mathrm{P}(\mathrm{z} - \mathrm{z_0}) = (\mathrm{z} - \mathrm{z_0})^{\mathrm{k}}[\mathrm{a_{k+1}} + \textstyle\sum_{\mathrm{n>k+1}} \mathrm{a_k}(\mathrm{z} - \mathrm{z_0})^{\mathrm{n}}]$$

At z_1, we only have one possible solution of ξ(z), and is zero, considering the above Taylor expression:

$$0 = (\mathrm{z_1} - \mathrm{z_0})^{\mathrm{k}}\mathrm{P}(\mathrm{z_1} - \mathrm{z_0}) = (\mathrm{z_1} - \mathrm{z_0})^{\mathrm{k}}[\mathrm{a_{k+1}} + \sum_{\mathrm{n>k+1}} \mathrm{a_k}(\mathrm{z_1} - \mathrm{z_0})^{\mathrm{n}}]$$

Taking modulus,

$$0 = |\varepsilon|^k\ |\mathrm{a_{k+1}} + \textstyle\sum_{\mathrm{n>k+1}} \mathrm{a_k}(\mathrm{z_1} - \mathrm{z_0})^{\mathrm{n}}|$$

According to properties of modulus and being in mind ε>0:

$$||\mathrm{a_{k+1}}| - \textstyle\sum_{\mathrm{n>k+1}} |\mathrm{a_k}||\varepsilon|^{\mathrm{n}}| \leq 0 \Rightarrow |\mathrm{a_{k+1}}| = \textstyle\sum_{\mathrm{n>k+1}} |\mathrm{a_k}||\varepsilon|^{\mathrm{n}} \qquad (1)$$

$$|e^{i\theta}\mathrm{a_{k+1}}| = |e^{i\theta}| \textstyle\sum_{\mathrm{n>k+1}} |\mathrm{a_k}||\varepsilon|^{\mathrm{n}} = \textstyle\sum_{\mathrm{n>k+1}} |\mathrm{e^{i\theta}a_k}||\varepsilon|^{\mathrm{n}} \Rightarrow a_{k+1} = -\textstyle\sum_{\mathrm{n>k+1}} \mathrm{e^{i\theta}a_k}(\mathrm{z_1} - \mathrm{z_0})^{\mathrm{n}} = \textstyle\sum_{\mathrm{n>k+1}} \mathrm{e^{i(\theta-\pi)}a_n}(\mathrm{z_1} - \mathrm{z_0})^{\mathrm{n}} \qquad (1^*)$$

Let's suppose there exists another z_2 with $\xi(z_2)=0$. Since the different $\{a_k\}$ are fixed -directly related to derivatives $\xi(z)$- the only thing varying is $(z_2 - z_0)^k$ but for each k, $|\varepsilon|^n$ remains, that would mean that if (1) is satisfied for a whatever z_2 in the border, it could be satisfied for all the frontier of $B(z_o, \varepsilon)$, and thus there would be an accumulated point at z_1). Impossible by the identity principle in analytic functions. In consequence, only in z_1, the function can be zero.

Summarizing, we´ve shown that out of Re(z)=1/2 we cannot find zeros of $\xi(z)$, thus zeros of $\Lambda(s)$, thus zeros of Riemman function $\varsigma(s)$. Together with Hardy's affirmation [2] the Riemann hypothese is demonstrated.

[1] E.Bombieri “Problems of the millennium: The Riemann hypothesis”. CLAY (2000).

[2] Hardy, G.H. Comptes Rendus, Acad. Sci. Paris, 158 (1914), 1012-1014.

[3] Kyrala, A: “Applied Functions of a complex variable”. Wiley-Interscience, N.York (1972).

[4] P.Sarnak Math. Acad. Sci., Paris 339, (2004), 91-94. 9.

[5] LMFDB Collaboration, \The Zeros of the Riemann Zeta-function. Database." http://www.lmfdb.org/zeros/zeta/, 2013. [Online; accessed 9 June 2018].